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UNCERTAINTIES IN MAXIMUM ENTROPY (ME) RECONSTRUCTIONS

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DISCLAIMER

UNCERTAINTIES IN MAXIMUM ENTROPY (ME) RECONSTRUCTIONS*

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This paper summarizes recent work done at the Lawrence Livermore National Laboratory by the writer on the effects of statistical uncertainty and image noise in Boltzmann ME inversion. The object of this work is the formulation of a Theory of Uncertainties which would allow one to compute confidence intervals for an object parameter near an ME reference value.

1. INTRODUCTION

The Boltzmann ME method resolves an object by a physical parameter such as density, photon density or power spectrum directly, without reference to an underlying probability density for the object parameters. At present, there appears to be no generally accepted comprehensive theory for accounting both for statistical uncertainty and measurement noise. Jaynes has stated [1] that a full Baysian solution is required, and cited the work of Gull and Daniell [2] as a step in that direction. Herein we offer our version of a "full Baysian solution."

2. STATISTICAL UNCERTAINTY

If the image (data) vector $\bar{T}$ is essentially noiseless (Fig. 1(a)) one most probable object $\bar{x}^{ME}$ is resolved. The statistical uncertainty of the object $\bar{x}$ around $\bar{x}^{ME}$ can be grossly described by the distribution of entropy $H(\bar{x})$ around H_{max} according to the Concentration Theorem [1]. The statistical uncertainty δx_i^S of any x_i around its x_i^{ME}-value as the number N of building blocks to realize an object approaches infinity is given by a Poisson distribution according to Darwin-Fowler (D-F) theory [3]. This result is independent of slight relaxation of the constraint relations between object and image and therefore will be valid with measurement noise present. The result also indicates we may treat object statistical uncertainty independent of object noise caused by measurement noise.

The D-F Theory is formulated in terms of integral numbers of building blocks of object parameter, such as density or photon intensity. It is not strictly appropriate for Boltzmann power spectrum analysis, for which the complex frequency or wave number amplitude usually does not have positive (or negative) definite real and imaginary components [4]. The precise description of statistical uncertainty in ME power spectrum analysis remains to be developed.

3. QUANTIZATION OF OBJECT VECTOR $\bar{x}$.

In order to have the distribution of $H(\bar{x})$ around H_{max} independent of scaling, and the

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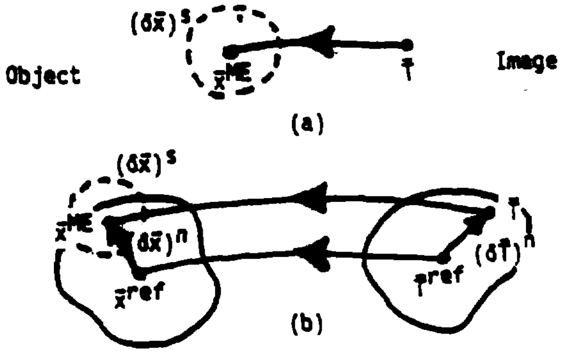

Figure 1. (a) Object $\bar{x}$ with statistical uncertainty $(\delta\bar{x})^s$ around $\bar{x}^{ME}$ determined by noiseless image $\bar{T}$. (b) Reference object $\bar{x}^{ref}$(ME) determined by reference image $\bar{T}^{ref}$. Each noisy vector $\bar{T} = \bar{T}^{ref} + (\delta\bar{T})^n$ yields a noisy object $\bar{x}^{ME} = \bar{x}^{ref} + (\delta\bar{x})^n$, with the superimposed statistical uncertainty $(\delta\bar{x})^s$.

Darwin-Fowler distribution independent of dimensioning, one must quantize $\bar{x}$; i.e., introduce a Δx (real) which is the limit of resolution in $\bar{x}$. This is true whether $\bar{T}$ is effectively noiseless or corrupted by measurement noise. Of course one need not quantize $\bar{x}$ in order to obtain an $\bar{x}^{ME}$, with or without measurement noise. The quantization is required by the D-F statistics, in terms of integral numbers of building blocks.

The philosophical implication of quantization is profound; it implies the statistical spread of object parameter around a reference value is narrower or broader depending on whether Δx is smaller or larger. Therefore, it is important to evaluate Δx realistically, according to the number of degrees of freedom in an object cell or measurement limitations, apart from limitations imposed by image noise.

The D-F theory yields the following expression for the statistical variance of δx_i^s in Figure 1(a),

$$\langle(\delta x_i^s)^2\rangle \;/\; (x_i^{ME})^2 = (\Delta x \;/\; x_i^{ME})\;(1-1/N) \qquad (1)$$

where N is the total number of building blocks in the object reconstruction. Usually N>>1. Clearly Δx will affect confidence intervals for x_i about a reference value x_i^{ref}.

4. IMAGE (MEASUREMENT) NOISE

Independent of statistical uncertainty discussed above, the image vector $\bar{T}$ is usually noisy around an ME reference value $\bar{T}^{ref}$ (Figure 1(b)). Each vector $\bar{T} = \bar{T}^{ref} + (\delta\bar{T})^n$ generates an ME object $\bar{x}^{ME} = \bar{x}^{ref} + (\delta\bar{x})^n$ with a probability distribution determined by that for $\delta\bar{T}^n$.

The ME inversion of the matrix constraint relation

$$\underset{\sim}{M}\bar{x}^{ME} = \bar{T}, \quad \underset{\sim}{M} = \underset{\sim}{M}(\bar{x}^{ME},\bar{T}) \qquad (2)$$

is the generalized inverse (for which we have a digital algorithm)

$$\bar{x}^{ME} = M^{-}(\bar{x}^{ME},\ \bar{T})\ \bar{T},$$
$$\bar{T} = T^{-ref} + (\delta\bar{T})^n \qquad (3)$$

This describes $\bar{x}^{ME}$ perturbed from $\bar{x}^{ref}$ by measurement noise $(\delta\bar{T})^n$ in Figure 1(b), apart from statistical uncertainty $(\delta\bar{x})^s$. The statistics of $\bar{x}^{ME}$ due to $(\delta\bar{T})^n$ are determined by

$$(\delta\bar{x})^n = \bar{x}^{ME} - \bar{x}^{ref} = \underset{\sim}{M}^{-}(\bar{x}^{ME},\ \bar{T})\ (\bar{T}^{ref} + \delta\bar{T}^n)$$
$$- \underset{\sim}{M}^{-}(\bar{x}^{ref},\ \bar{T}^{ref})\ \bar{T}^{ref} \qquad (4)$$

For small excursions, the object noise can be approximated as

$$(\delta\bar{x})^n = \underset{\sim}{M}^{-}(\bar{x}^{ref},\bar{T}^{ref})\ (\delta\bar{T})^n \qquad (5)$$

We propose defining the assurance A_i that $(\delta x_i)^n$ is smaller than a specified number a_i as

$$A_i(a_i) = \mathrm{Prob}\left[|(\delta x_i)^n| \le a_i\right] \qquad (6)$$

computable from the statistics of $(\delta\bar{T})^n$.

5. TOTAL UNCERTAINTY IN OBJECT PARAMETER, x_i

According to Figure 1(b), the object vector $\bar{x}$ is perturbed from $\bar{x}^{ref}$ by image noise and statistical uncertainty,

$$x_i = x_i^{ref} + (\delta x_i)^n + (\delta x_i)^s \ .$$

The total variance of x_i is, for small image noise where (5) is appropriate,

$$\langle(x_i - x_i^{ref})^2\rangle = \sum_{k\ell} M^-_{ik} M^-_{i\ell} \langle\delta T_k \delta T_\ell\rangle + \langle(\delta x_i^s)^2\rangle \qquad (7)$$

Either (4) or (5) and the Poisson distribution for $(\delta x_i)^s$ would determine a confidence C_i that x_i lay within a specified interval c_i defined as

$$C_i(c_i) = \mathrm{Prob}\left[|x_i - x_i^{ref}| \le c_i\right] \qquad (8)$$

6. CONCLUSIONS

The Theory of Uncertainties just outlined appears to be practical computationally. The statistical uncertainty is dependent on the object parameter quantization, Δx, which must be realistically chosen. It is reasonable to define an assurance that an object parameter is close to its reference value according to measurement (image) noise alone and a confidence that the parameter is close to its reference value according to both noise and statistical uncertainty.

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